A comment on the normal & osculating planes:

Recall that we only need a vector which is perpendicular to the plane to find an equation for it, in particular, the length of the vector doesn't matter. So, easier vectors to find to use are:

<u>Normal Plane</u>: use $\vec{r}\,'(t)$

<u>Osculating Plane</u>: use $\vec{r}\,'(t) \times \vec{r}\,''(t)$

Lecture 7

13.4 - Motion in Space

Suppose a particle moves along a trajectory $\vec{r}(t)$.

Its <u>velocity</u> is $\vec{v}(t) = \vec{r}\,'(t)$,

<u>acceleration</u> is $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$,

and <u>speed</u> is $\|\vec{v}(t)\|$, which I will denote by v.

<u>Ex</u>: A particle has acceleration function

$$\vec{a}(t) = 4t\,\hat{i} + 6\sin t\,\hat{j} + e^t\,\hat{k}.$$

If its initial velocity is $\vec{v}(0) = 3\hat{j}$ and its initial position is $\vec{r}(0) = \vec{0}$, find its position function.

<u>Sol</u>: $\vec{v}(t) = \int \vec{a}(t)\,dt = (2t^2 + C_1)\hat{i} + (-6\cos t + C_2)\hat{j} + (e^t + C_3)\hat{k}$

$\vec{v}(0) = C_1\hat{i} + (-6 + C_2)\hat{j} + (1 + C_3)\hat{k} = 3\hat{j} \Rightarrow C_1 = 0,\ C_2 = 9,\ C_3 = -1$

$\Rightarrow \vec{v}(t) = 2t^2\hat{i} + (9 - 6\cos t)\hat{j} + (e^t - 1)\hat{k}$

Now, $\vec{r}(t) = \int \vec{v}(t)\,dt = \left(\frac{2}{3}t^3 + D_1\right)\hat{\imath} + (9t - 6\sin t + D_2)\hat{\jmath} + (e^t - t + D_3)\hat{k}$

$\vec{r}(0) = D_1\hat{\imath} + D_2\hat{\jmath} + (1 + D_3)\hat{k} = \vec{0} \Rightarrow D_1 = D_2 = 0,\ D_3 = -1$

So, $\vec{r}(t) = \frac{2}{3}t^3\hat{\imath} + (9t - 6\sin t)\hat{\jmath} + (e^t - t - 1)\hat{k}$

$\diamond$

If the particle has mass m and acceleration $\vec{a}(t)$, the force it experiences is given by <u>Newton's second law</u>:

$$\vec{F}(t) = m\vec{a}(t).$$

<u>Ex</u>: A projectile is fired with muzzle speed 200 m/s and angle of elevation 60°. If the projectile is fired from a distance of 10 m above ground level, what is the distance covered by the projectile? (All forces, except gravity, are assumed negligable.)

<u>Sol</u>: The only force acting on the projectile is gravity, so $\vec{F}(t) = m\vec{a}(t) = -mg\hat{\jmath}$. This means $\vec{a}(t) = -g\hat{\jmath}$.

So, $\vec{v}(t) = \int \vec{a}(t)\,dt = C_1\hat{\imath} + (-gt + C_2)\hat{\jmath}$

To get the initial velocity, we use the given information $\vec{v}_0 = \vec{v}(0)$ & $v_0 = \|\vec{v}_0\|$. Then $v_0 = 200$, so

$\vec{v}_0 = (200\cos 60°)\hat{\imath} + (200\sin 60°)\hat{\jmath} = 100\hat{\imath} + 100\sqrt{3}\,\hat{\jmath}$

$\Rightarrow C_1 = 100$ & $C_2 = 100\sqrt{3}$

Thus the velocity function is:

$$\vec{v}(t) = 100\hat{i} + (100\sqrt{3} - gt)\hat{j}$$

The position function is:

$$\vec{r}(t) = \int \vec{v}(t)\,dt = (100t + D_1)\hat{i} + (100\sqrt{3}\,t - \tfrac{1}{2}gt^2 + D_2)\hat{j}$$

The initial position is $\vec{r}(0) = 10\hat{j}$, so $D_1 = 0, D_2 = 10$.

Thus: $\vec{r}(t) = 100t\,\hat{i} + (100\sqrt{3}\,t - \frac{1}{2}gt^2 + 10)\hat{j}$.

The particle hits the ground when the $\hat{j}$-component is 0:

$$100\sqrt{3}\,t - \tfrac{1}{2}gt^2 + 10 = 0 \Rightarrow t = \frac{-100\sqrt{3} \pm \sqrt{30000 + 20g}}{-g}$$

We take the positive value of t:

$$t = \frac{-100\sqrt{3} - \sqrt{30000 + 20g}}{-g} \overset{(g \approx 9.8)}{\approx} 35.4$$

Plugging this in the $\hat{i}$-component gives the distance traveled: dist $= 100(35.4)\,m = 3540\,m$. ◇

Recall that the motion of a curve is best captured by the osculating plane at any point. (After all, $\vec{B}(t) \perp \vec{r}\,'(t), \vec{r}\,''(t)$.) We aim to write the acceleration in terms of $\vec{T}(t)$ and $\vec{N}(t)$.

Let's start with $\vec{T}$:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\vec{v}(t)}{\nu(t)} \Rightarrow \vec{v}(t) = \nu(t)\vec{T}(t)$$

Take a derivative:

$$\vec{v}\,'(t) = \nu'(t)\vec{T}(t) + \nu(t)\vec{T}\,'(t) = \vec{a}(t).$$

Now, $\kappa(t) = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \dfrac{\|\vec{T}\,'(t)\|}{\nu(t)} \Rightarrow \|\vec{T}\,'(t)\| = \nu(t)\kappa(t)$

This allows us to write:

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{\vec{T}\,'(t)}{\nu(t)\kappa(t)} \Rightarrow \vec{T}\,'(t) = \nu(t)\kappa(t)\vec{N}(t)$$

Finally: $\vec{a}(t) = \nu'(t)\vec{T}(t) + (\nu(t))^2\kappa(t)\vec{N}(t)$

This motivates the definitions:

<u>tangential component of acceleration</u>: $\boxed{a_T = \nu'}$

<u>normal " " "</u>: $\boxed{a_N = \nu^2\kappa}$

With a little work, one can find:

$$\boxed{a_T = \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\vec{r}\,'(t)\|} \quad \& \quad a_N = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|}}$$

A convenient fact:

$\vec{a} = a_T \vec{T} + a_N \vec{N}$. Since $\vec{T} \cdot \vec{N} = 0$ & $\vec{T} \cdot \vec{T} = \vec{N} \cdot \vec{N} = 1$:

$$\|\vec{a}\|^2 = \vec{a} \cdot \vec{a} = (a_T \vec{T} + a_N \vec{N}) \cdot (a_T \vec{T} + a_N \vec{N}) = a_T^{\,2} + a_N^{\,2}$$

So, $\|\vec{a}\| = \sqrt{a_T^{\,2} + a_N^{\,2}}$

Ex: Find the normal and tangential components of acceleration for a particle moving along the trajectory $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Sol: $\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\vec{r}''(t) = \langle -\cos t, -\sin t, 0 \rangle$

$$\vec{r}'(t) \cdot \vec{r}''(t) = \sin t \cos t - \cos t \sin t + 0 = 0$$

$$\Rightarrow a_T = 0 \Rightarrow \|\vec{a}\| = \sqrt{a_T^{\,2} + a_N^{\,2}} = \sqrt{a_N^2} = a_N$$

$$a_N = \|\vec{a}(t)\| = \|\vec{r}''(t)\| = \sqrt{\cos^2 t + \sin^2 t + 0^2} = 1$$

◇

14.1 - Functions of Several Variables

Functions of Two Variables

A function of two variables, f, is a rule which assigns to each pair (x,y) in a set $D \subset \mathbb{R}^2$ a unique real number $f(x,y)$. D is called the domain of f and the range of f is the set of values it takes on, i.e., $\{f(x,y) \mid (x,y) \in D\}$.

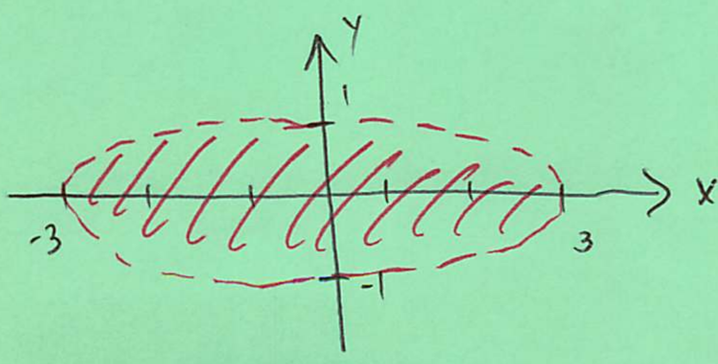
If we write $z = f(x,y)$, x & y are the independent variables and z is the dependent variable.

Ex: Find the domain of $f(x,y) = \ln(9 - x^2 - 9y^2)$ and sketch it.

Sol: Recall that $\ln(t)$ is defined for $t > 0$

So, we need $9 - x^2 - 9y^2 > 0 \Leftrightarrow x^2 + 9y^2 < 9 \Leftrightarrow \frac{x^2}{9} + y^2 < 1$

The domain is $D = \{(x,y) \mid \frac{x^2}{9} + y^2 < 1\}$ and a sketch is



$\Diamond$

Ex: Sketch the graph of $h(x,y)=4x^2+y^2$.

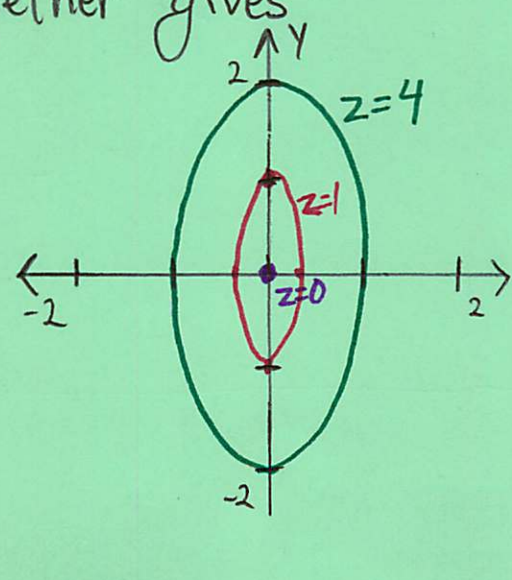
Sol: The graph is the set of points $(x,y,z)\in\mathbb{R}^3$ such that $z=h(x,y)$. We want to graph $z=4x^2+y^2$. We will see what this looks like using level curves, that is, taking slices of the graph at different z-values. Note that $z\geq 0$ here.

$z=0$: $0=4x^2+y^2 \Rightarrow x=y=0$. So, the level curve for $z=0$ is the origin $(0,0)$

$z=1$: $1=4x^2+y^2 \Leftrightarrow \frac{x^2}{(\frac{1}{2})^2}+y^2=1$. So, the level curve at height $z=1$ is an ellipse.

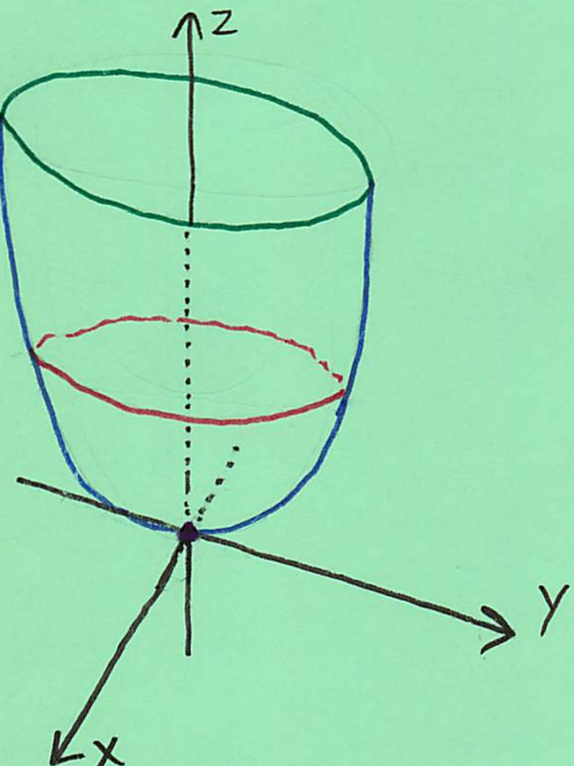
$z=4$: $4=4x^2+y^2 \Leftrightarrow x^2+\frac{y^2}{4}=1$. An ellipse again.

Putting these together gives:



The way to read this graph is to realize that the level curve you sit on determines your height.

Lifting the level curves to their respective z-values, we see the elliptic paraboloid we expect:



◇

The 2D picture is called the contour map of h. The level curves are sometimes called contours. You would see this kind of plot when you look at temperature maps or topography maps.

Functions of Three or More Variables

A function, f, of three variables will take in a triple (x,y,z) and output a real number $f(x,y,z)$. The set of allowable input points (x,y,z) is its domain. Notice that we cannot graph a function of 3 variables since its graph is 4 dimensional!

The best we can do is study "pictures of it in time", i.e., sets $f(x,y,z)=c$. In this case they are called <u>level surfaces</u>.

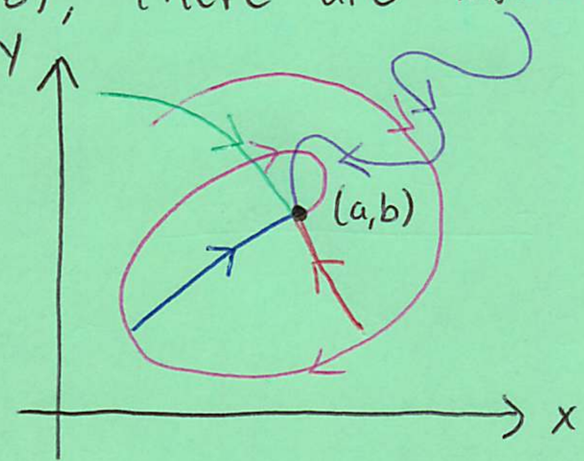
For functions of more than 3 variables, sets of points $(x_1, \cdots, x_n, c)$ are called <u>level sets</u> of the function. They're going to be impossible to plot.

<u>14.2</u> - Limits and Continuity

<u>Def</u>: Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the <u>limit of $f(x,y)$ as (x,y) approaches (a,b)</u> is L if for every $\varepsilon>0$ there is a $\delta>0$ such that if $(x,y)\in D$ and $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$, then $|f(x,y)-L|<\varepsilon$. If this is so, we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Intuitively, this means as the inputs of f get closer to (a,b), the values f can take on are "squeezed towards L". These limits are exceptionally hard to compute, in general. This is because, in $\mathbb{R}^2$, there are more than just two paths to (a,b), there are infinitely many! Examples are:



Based on this, it's easier usually to show limits don't exist. This is because showing that the limit along two different paths are not equal is easier than showing all of the limits along all of the infinitely many paths are the same.

Ex: Does the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exist?

Sol: Let travel along the positive x-axis:

this means we use points $(x,0)$, $x>0$ and let $x\to 0$:

$$\lim_{(x,0)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

now, along the positive y-axis:

$$\lim_{(0,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

$1\neq -1$, so the limit does not exist! ◇

Be careful though, this x-&y-axis trick doesn't always work. For example:

Ex: Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ exist?

Sol: If we plug in the x- or y-axis, the top becomes zero, so the limits along the axes are zero.

However, along $y=x$:

$$\lim_{(x,x)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$$

So, the limit does not exist. ◇

Lecture 8

A convenient trick to use to show limits DO NOT exist is to check along the line $y=mx$, where m is arbitrary. (This is for limits to $(0,0)$.) If there is a dependence on m, the limit does not exist. Consider again the last example. Along $y=mx$ we have

$$\lim_{(x,mx)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

since m is arbitrary, this limit does not exist.

This trick, however, is not cure-all.

Ex: Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ exist?

Sol: along $y=mx$: $\lim\limits_{(x,mx)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = \lim\limits_{x\to 0} \dfrac{m^2x^3}{x^2+m^4x^4} = \lim\limits_{x\to 0} \dfrac{m^2x}{1+m^4x^2} = 0.$